4·2. PARTIAL DERIVATIVES

Let $z = f(x, y)$ be a function of two variables x and y.

If we keep y as constant and vary x alone, then z is a function of x only. The derivative of z with respect to x, treating y as constant, is called the *partial derivative of z with respect to x* and is denoted by one of the symbols

$$\frac{\partial z}{\partial x}, \frac{\partial f}{\partial x}, f_x(x, y), D_x f. \qquad \text{Thus} \quad \frac{\partial z}{\partial x} = \underset{\delta x \to 0}{\text{Lt}} \frac{f(x + \delta x, y) - f(x, y)}{\delta x}$$

Similarly, the derivative of z with respect to y, keeping x as constant, is called the *partial derivative of z with respect to y* and is denoted by one of the symbols

$$\frac{\partial z}{\partial y}, \frac{\partial f}{\partial y}, f_y(x, y), D_y f. \qquad \text{Thus} \quad \frac{\partial z}{\partial y} = \underset{\delta y \to 0}{\text{Lt}} \frac{f(x, y + \delta y) - f(x, y)}{\delta y}$$

Similarly, if z is a function of three or more variables $x_1, x_2, x_3, \ldots$ *the partial derivative of z with respect to x_1, is obtained by differentiating z with respect to x_1, keeping all other variables constant* and is written as $\partial z / \partial x_1$.

In general f_x and f_y are also functions of x and y and so these can be differentiated further partially with respect to x and y.

Thus $$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2} \text{ or } \frac{\partial^2 f}{\partial x^2} \text{ or } f_{xx}, \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial y} \text{ or } \frac{\partial^2 f}{\partial x \partial y} \text{ or } f_{yx}^*$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y \partial x} \text{ or } \frac{\partial^2 f}{\partial y \partial x} \text{ or } f_{xy}, \text{ and } \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2} \text{ or } \frac{\partial^2 f}{\partial y^2} \text{ or } f_{yy}.$$

It can easily be verified that, in all ordinary cases,

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}.$$

Example 4·1. *Find the first and second partial derivatives of $z = x^3 + y^3 - 3axy$.*

We have $z = x^3 + y^3 - 3axy$.

$\therefore$ $$\frac{\partial z}{\partial x} = 3x^2 + 0 - 3ay(1) = 3x^2 - 3ay, \text{ and } \frac{\partial z}{\partial y} = 0 + 3y^2 - 3ax(1) = 3y^2 - 3ax$$

Also $$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}(3x^2 - 3ay) = 6x, \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}(3x^2 - 3ay) = -3a$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}(3y^2 - 3ax) = 6y, \text{ and } \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}(3y^2 - 3ax) = -3a.$$

We observe that $\dfrac{\partial^2 z}{\partial y \partial x} = \dfrac{\partial^2 z}{\partial x \partial y}$.

Example 4·2. *If* $u = x^2 \tan^{-1}\dfrac{y}{x} - y^2 \tan^{-1}\dfrac{x}{y}$,

show that $$\frac{\partial^2 u}{\partial x \partial y} = \frac{x^2 - y^2}{x^2 + y^2}.$$ *(Mysore, 1997 ; Marathwada, 1994)*

and $$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$$ *(Madras, 1999 ; A.M.I.E., 1994)*

We have $$\frac{\partial u}{\partial y} = x^2 \cdot \frac{1}{1 + (y/x)^2} \cdot \frac{1}{x} - \left\{2y \cdot \tan^{-1}\frac{x}{y} + y^2 \cdot \frac{1}{1 + (x/y)^2} \cdot \left(-\frac{x}{y}\right)\right\}$$

---

* **It is important to note that in the subscript notation the subscripts are written in the same order in which we differentiate whereas in the '∂' notation the order is opposite.**

Scanned with CamScanner

$$= \frac{x^3}{x^2+y^2} - 2y \tan^{-1}\frac{x}{y} + \frac{xy^2}{x^2+y^2} = x - 2y \tan^{-1}\frac{x}{y}.$$

$$\therefore \quad \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left\{x - 2y \tan^{-1}\frac{x}{y}\right\} = 1 - 2y \cdot \frac{1}{1+(x/y)^2} \cdot \frac{1}{y} = 1 - \frac{2y^2}{x^2+y^2} = \frac{x^2-y^2}{x^2+y^2}.$$

Similarly, $\frac{\partial u}{\partial x} = 2x \tan^{-1} y/x - y$

and $$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left\{2x \tan^{-1}\frac{y}{x} - y\right\} = \frac{x^2-y^2}{x^2+y^2}.$$ Hence the result.

Example 4·3. *If* $z = f(x+ct) + \phi(x-ct)$, *prove that*

$$\frac{\partial^2 z}{\partial t^2} = c^2 \frac{\partial^2 z}{\partial x^2}.$$

We have $$\frac{\partial z}{\partial x} = f'(x+ct) \cdot \frac{\partial}{\partial x}(x+ct) + \phi'(x-ct)\frac{\partial}{\partial x}(x-ct) = f'(x+ct) + \phi'(x-ct)$$

and $$\frac{\partial^2 z}{\partial x^2} = f''(x+ct) + \phi''(x-ct) \qquad \ldots(i)$$

Again $$\frac{\partial z}{\partial t} = f'(x+ct)\frac{\partial}{\partial t}(x+ct) + \phi'(x-ct)\frac{\partial}{\partial t}(x-ct) = cf'(x+ct) - c\phi'(x-ct)$$

and $$\frac{\partial^2 z}{\partial t^2} = c^2 f''(x+ct) + c^2\phi''(x-ct) = c^2[f''(x+ct) + \phi''(x-ct)] \qquad \ldots(ii)$$

From (i) and (ii), it follows that $\frac{\partial^2 z}{\partial t^2} = c^2 \frac{\partial^2 z}{\partial x^2}$.

Obs. This is an important partial differential equation, known as *wave equation* (§ 17·4).

Example 4·4. *If* $v = (x^2+y^2+z^2)^{-1/2}$, *prove that*

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = 0. \text{ (Laplace equation)*}$$

(Mangalore, 1997 ; S. Gujarat, 1990)

We have $$\frac{\partial v}{\partial x} = -\frac{1}{2}(x^2+y^2+z^2)^{-3/2} \cdot 2x = -x(x^2+y^2+z^2)^{-3/2}$$

and $$\frac{\partial^2 v}{\partial x^2} = -[1 \cdot (x^2+y^2+z^2)^{-3/2} + x(-3/2)(x^2+y^2+z^2)^{-5/2} \cdot 2x]$$

$$= -(x^2+y^2+z^2)^{-5/2}[x^2+y^2+z^2-3x^2] = (x^2+y^2+z^2)^{-5/2}(2x^2-y^2-z^2)$$

Similarly $\frac{\partial^2 v}{\partial y^2} = (x^2+y^2+z^2)^{-5/2}(-x^2+2y^2-z^2)$ and $\frac{\partial^2 v}{\partial z^2} = (x^2+y^2+z^2)^{-5/2}(-x^2-y^2+2z^2)$

Hence $$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = (x^2+y^2+z^2)^{-5/2} \cdot (0) = 0.$$

Obs. *A function v satisfying the Laplace equation* is said to be a* **harmonic function.**

Example 4·5. *If* $u = x^y$, *show that* $\frac{\partial^3 u}{\partial x^2 \partial y} = \frac{\partial^3 u}{\partial x \partial y \partial x}$.

We have $\frac{\partial u}{\partial y} = x^y \log_e x$ and $\frac{\partial^2 u}{\partial x \partial y} = yx^{y-1} \cdot \log x + x^y \cdot \frac{1}{x} = x^{y-1}(y \log x + 1)$

* See footnote p. 27.

Scanned with CamScanner

Now to find $\frac{\partial r}{\partial x}, \frac{\partial r}{\partial y}$ etc., we write $r = (x^2 + y^2)^{1/2}$

$$\frac{\partial r}{\partial x} = \frac{1}{2}(x^2 + y^2)^{-1/2} \cdot 2x = \frac{x}{r}$$

$\therefore$

$$\frac{\partial^2 r}{\partial x^2} = \frac{r \cdot 1 - x \cdot \partial r/\partial x}{r^2} = \frac{r - x^2/r}{r^2} = \frac{y^2}{r^3}$$

and

Similarly $\frac{\partial r}{\partial y} = \frac{y}{r}$ and $\frac{\partial^2 r}{\partial y^2} = \frac{x^2}{r^3}$.

Substituting the values of $\partial r/\partial x$ etc. in (i), we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r)\left[\frac{x^2}{r^2} + \frac{y^2}{r^2}\right] + f'(r)\left[\frac{y^2}{r^3} + \frac{x^2}{r^3}\right] = f''(r) + \frac{1}{r}f'(r).$$

Problems 4·3

1. If $x = r\cos\theta, y = r\sin\theta$, show that $\frac{\partial r}{\partial x} = \frac{\partial x}{\partial r}$ and $\frac{1}{r}\frac{\partial x}{\partial \theta} = r\frac{\partial \theta}{\partial x}$.

2. If $x^2 = au + bv, y^2 = au - bv$, prove that $\left(\frac{\partial u}{\partial x}\right)_y \cdot \left(\frac{\partial x}{\partial u}\right)_v = \frac{1}{2} = \left(\frac{\partial v}{\partial y}\right)_x \cdot \left(\frac{\partial y}{\partial v}\right)_u$.

3. If $u = lx + my, v = mx - ly$, show that $\left(\frac{\partial u}{\partial x}\right)_y \left(\frac{\partial x}{\partial u}\right)_v = \frac{l^2}{l^2 + m^2}, \left(\frac{\partial y}{\partial v}\right)_x \left(\frac{\partial v}{\partial y}\right)_u = \frac{l^2 + m^2}{l^2}$. *(Marathwada, 1990)*

4. If $x = r\cos\theta, y = r\sin\theta$, prove that (i) $\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} = \frac{1}{r}\left[\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2\right]$. *(Mysore, 1994 S)*

(ii) $\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} = 0$ $(x \neq 0, y \neq 0)$.

5. If $u = f(r)$ where $r = \sqrt{(x^2 + y^2 + z^2)}$, show that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = f''(r) + \frac{2}{r}f'(r)$.

4·4. (1) HOMOGENEOUS FUNCTIONS

An expression of the form

$$a_0x^n + a_1x^{n-1}y + a_2x^{n-2}y^2 + \ldots\ldots + a_ny^n$$

in which every term is of the nth degree, is called a homogeneous function of degree n. This can be rewritten as

$$x^n\,[a_0 + a_1(y/x) + a_2(y/x)^2 + \ldots\ldots + a_n\,(y/x)^n].$$

Thus *any function $f(x, y)$ which can be expressed in the form $x^n\,\phi\,(y/x)$, is called a* **homogeneous function** of degree n in x and y.

For instance, $x^3\cos(y/x)$ is a homogeneous function of degree 3, in x and y.

In general, a function $f(x, y, z, t, \ldots\ldots)$ is said to be a homogeneous function of degree n in x, y, z, t,....., if it can be expressed in the form $x^n\phi\,(y/x, z/x, t/x, \ldots\ldots)$.

(2) Euler's theorem on homogeneous functions.* *If u be a homogeneous function of degree n in x and y, then*

$$\mathbf{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu.}$$

* See footnote p. 90.

Scanned with CamScanner

Since u is a homogeneous function of degree n in x and y, therefore,

$$u = x^n f(y/x)$$

$$\therefore \quad \frac{\partial u}{\partial x} = nx^{n-1} f\left(\frac{y}{x}\right) + x^n f'\left(\frac{y}{x}\right) \cdot y\left(-\frac{1}{x^2}\right) = nx^{n-1} f\left(\frac{y}{x}\right) - yx^{n-2} f'\left(\frac{y}{x}\right)$$

and $\quad \frac{\partial u}{\partial y} = x^n f'\left(\frac{y}{x}\right) \cdot \frac{1}{x} = x^{n-1} f'\left(\frac{y}{x}\right)$. Hence $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nx^n f\left(\frac{y}{x}\right) = nu$.

In general, if u be a homogeneous function of degree n in $x, y, z, t, \ldots\ldots$, then,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} + t\frac{\partial u}{\partial t} \ldots\ldots = nu.$$

Example 4.7. *Show that* $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u \log u$ *where* $\log u = (x^3 + y^3)/(3x + 4y)$.

(Mysore, 1998)

Since $z = \log u = \frac{x^3 + y^3}{3x + 4y} = x^2 \cdot \frac{1 + (y/x)^3}{3 + 4(y/x)}$,

$\therefore$ z is a homogeneous function of degree 2 in x and y.

By Euler's theorem, we get

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2z \qquad \ldots(i)$$

But $\quad \frac{\partial z}{\partial x} = \frac{1}{u}\frac{\partial u}{\partial x}$ and $\frac{\partial z}{\partial y} = \frac{1}{u}\frac{\partial u}{\partial y}$

Hence (i) becomes

$$x \cdot \frac{1}{u}\frac{\partial u}{\partial x} + y \cdot \frac{1}{u}\frac{\partial u}{\partial y} = 2 \log u \quad \text{or} \quad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u \log u$$

Example 4.8. *If* $u = \sin^{-1}\frac{x + 2y + 3z}{x^8 + y^8 + z^8}$, *find the value of* $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$.

(Marathwada, 1998)

Here u is not a homogeneous function. We, therefore, write

$$\omega = \sin u = \frac{x + 2y + 3z}{x^8 + y^8 + z^8} = x^{-7} \cdot \frac{1 + 2(y/x) + 3(z/x)}{1 + (y/x)^8 + (z/x)^8}$$

Thus ω is a homogeneous function of degree -7 in x, y, z. Hence by Euler's theorem

$$x\frac{\partial \omega}{\partial x} + y\frac{\partial \omega}{\partial y} + z\frac{\partial \omega}{\partial z} = (-7)\,\omega \qquad \ldots(i)$$

But $\quad \frac{\partial \omega}{\partial x} = \cos u \frac{\partial u}{\partial x}$, $\frac{\partial \omega}{\partial y} = \cos u \frac{\partial u}{\partial y}$, $\frac{\partial \omega}{\partial z} = \cos u \frac{\partial u}{\partial z}$

$\therefore$ (i) becomes $\quad x\cos u\frac{\partial u}{\partial x} + y\cos u\frac{\partial u}{\partial y} + z\cos u\frac{\partial u}{\partial z} = -7\sin u$

or $\quad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = -7\tan u$.

Example 4.9. *If* z *is a homogeneous function of degree* n *in* x *and* y, *show that*

$$x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x \partial y} + y^2\frac{\partial^2 z}{\partial z^2} = n(n-1)z.$$

(Mangalore, 1997 ; C.S.E., 1995)

By Euler's theorem, $\quad x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz \qquad \ldots(i)$

Scanned with CamScanner

Differentiating (i) partially w.r.t. x, we get $x\dfrac{\partial^2 z}{\partial x^2}+\dfrac{\partial z}{\partial x}+y\dfrac{\partial^2 z}{\partial x\partial y}=n\dfrac{\partial z}{\partial x}$

$$x\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial x\partial y}=(n-1)\frac{\partial z}{\partial x} \qquad \text{...(ii)}$$

Again differentiating (i) partially w.r.t. y, we get $x\dfrac{\partial^2 z}{\partial y\partial x}+\dfrac{\partial z}{\partial y}+y\dfrac{\partial^2 z}{\partial y^2}=n\dfrac{\partial z}{\partial y}$

$$x\frac{\partial^2 z}{\partial x\partial y}+y\frac{\partial^2 z}{\partial y^2}=(n-1)\frac{\partial z}{\partial y} \qquad \text{...(iii)}$$

Multiplying (ii) by x and (iii) by y and adding, we get

$$x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\partial y}+y^2\frac{\partial^2 z}{\partial y^2}=(n-1)\left(x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}\right)=n(n-1)z \qquad \text{[by (i)}$$

Problems 4.4

1. Show that $f(x, y, z)=(4x^2+2y^2z)/(x+2y+3z)$ is homogeneous. What is its degree ?

2. Verify Eulers's theorem, when (i) $f(x, y)=ax^2+2hxy+by^2$ (*Osmania, 1995*)

   (ii) $f(x, y)=x^2(x^2-y^2)^2/(x^2+y^2)^2$ (*Bhopal, 1991*)

   (iii) $f(x, y)=3x^2yz+5xy^2z+4z^3$ (*J.N.T.U., 1999*)

3. If $u=\sin^{-1}\dfrac{x}{y}+\tan^{-1}\dfrac{y}{x}$, prove that $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=0$. (*Mysore, 1994 S ; J.N.T.U., 1990*)

4. If $u=\sin^{-1}\dfrac{x^3+y^3}{x+y}$, prove that $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=\tan u$ (*J.N.T.U., 1999 ; Mangalore, 1997*)

5. If $\sin u=\dfrac{x^3y^3}{x+y}$, show that $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=3\tan u$. (*Karnataka, 1993*)

6. If $u=\cos^{-1}\dfrac{x+y}{\sqrt{x}+\sqrt{y}}$, prove that $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=-\dfrac{1}{2}\cot u$. (*Gorakhpur, 1991*)

7. If $z=\log(x^3+xy+y^3)$, show that $x\dfrac{\partial z}{\partial x}+y\dfrac{\partial z}{\partial y}=3$. (*Gauhati, 1999*)

8. (a) If u is homogeneous function of nth degree in x, y, z, prove that $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}+z\dfrac{\partial u}{\partial z}=nu$. (*N. Bengal, 1989*)

   (b) Show that $xu_x+yu_y+zu_z=2\tan u$, where $u=\sin^{-1}\left(\dfrac{x^3+y^3+z^3}{ax+by+cz}\right)$ (*Mysore, 1997 S*)

9. If $z=xu\left(\dfrac{y}{x}\right)+w\left(\dfrac{y}{x}\right)$, prove that $x^2\dfrac{\partial^2 z}{\partial x^2}+2xy\dfrac{\partial^2 z}{\partial x\partial y}+y^2\dfrac{\partial^2 z}{\partial y^2}=0$. (*Kavempu, 1998 S; A.M.I.E., 1997 W*)

10. If $u=\tan^{-1}\dfrac{x^3+y^3}{x-y}$, prove that $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=\sin 2u$. (*Mysore, 1996*)

    and $x^2\dfrac{\partial^2 u}{\partial x^2}+2xy\dfrac{\partial^2 u}{\partial x\partial y}+y^2\dfrac{\partial^2 u}{\partial y^2}=2\cos 2u\sin u$. (*Assam, 1998; J.N.T.U., 1998; Ranchi, 1998*)

11. Given $x=x^4f_1(y/x)+y^{-4}f_2(x/y)$, prove that $x^2\dfrac{\partial^2 z}{\partial x^2}+2xy\dfrac{\partial^2 z}{\partial x\partial y}+y^2\dfrac{\partial^2 z}{\partial y^2}+x\dfrac{\partial z}{\partial x}+y\dfrac{\partial z}{\partial y}=n^2z$. (*Marathwada, 1994*)

12. If $u=x^3\tan^{-1}(y/x)-y^3\tan^{-1}(x/y)$, evaluate $x^2\dfrac{\partial^2 u}{\partial x^2}+2xy\dfrac{\partial^2 u}{\partial x\partial y}+y^2\dfrac{\partial^2 u}{\partial y^2}$ (*Bangalore, 1990*)

Scanned with CamScanner